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APPLICATION NO.	FILING DATE	FIRST NAMED INVENTOR	ATTORNEY DOCKET NO.	CONFIRMATION NO.
10/713,450	11/14/2003	Oscar E. Agazzi	13449US05	9631

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EXAMINER	
GHEBRETINSAE, TEMESGHEN	
ART UNIT	**PAPER NUMBER**
2637	

DATE MAILED: 12/19/2005

Please find below and/or attached an Office communication concerning this application or proceeding.

Office Action Summary	Application No.	Applicant(s)
	10/713,450	AGAZZI, OSCAR E.
	Examiner	**Art Unit**
	Temesghen Ghebretinsae	2637

-- The MAILING DATE of this communication appears on the cover sheet with the correspondence address --

Period for Reply

A SHORTENED STATUTORY PERIOD FOR REPLY IS SET TO EXPIRE 3 MONTH(S) OR THIRTY (30) DAYS, WHICHEVER IS LONGER, FROM THE MAILING DATE OF THIS COMMUNICATION.

- Extensions of time may be available under the provisions of 37 CFR 1.136(a). In no event, however, may a reply be timely filed after SIX (6) MONTHS from the mailing date of this communication.
- If NO period for reply is specified above, the maximum statutory period will apply and will expire SIX (6) MONTHS from the mailing date of this communication.
- Failure to reply within the set or extended period for reply will, by statute, cause the application to become ABANDONED (35 U.S.C. § 133). Any reply received by the Office later than three months after the mailing date of this communication, even if timely filed, may reduce any earned patent term adjustment. See 37 CFR 1.704(b).

Status

1)☒ Responsive to communication(s) filed on 28 July 2005.

2a)☐ This action is **FINAL**. 2b)☒ This action is non-final.

3)☐ Since this application is in condition for allowance except for formal matters, prosecution as to the merits is closed in accordance with the practice under *Ex parte Quayle*, 1935 C.D. 11, 453 O.G. 213.

Disposition of Claims

4)☒ Claim(s) 24-33 is/are pending in the application.

4a) Of the above claim(s) _____ is/are withdrawn from consideration.

5)☒ Claim(s) 21,25,29 and 33 is/are allowed.

6)☒ Claim(s) 24,26-28 and 32 is/are rejected.

7)☐ Claim(s) _____ is/are objected to.

8)☐ Claim(s) _____ are subject to restriction and/or election requirement.

Application Papers

9)☐ The specification is objected to by the Examiner.

10)☐ The drawing(s) filed on _____ is/are: a)☐ accepted or b)☐ objected to by the Examiner.

Applicant may not request that any objection to the drawing(s) be held in abeyance. See 37 CFR 1.85(a).

Replacement drawing sheet(s) including the correction is required if the drawing(s) is objected to. See 37 CFR 1.121(d).

11)☐ The oath or declaration is objected to by the Examiner. Note the attached Office Action or form PTO-152.

Priority under 35 U.S.C. § 119

12)☐ Acknowledgment is made of a claim for foreign priority under 35 U.S.C. § 119(a)-(d) or (f).

a)☐ All b)☐ Some * c)☐ None of:

1.☐ Certified copies of the priority documents have been received.

2.☐ Certified copies of the priority documents have been received in Application No. _____.

3.☐ Copies of the certified copies of the priority documents have been received in this National Stage application from the International Bureau (PCT Rule 17.2(a)).

* See the attached detailed Office action for a list of the certified copies not received.

Attachment(s)

1) ☐ Notice of References Cited (PTO-892)
2) ☐ Notice of Draftsperson's Patent Drawing Review (PTO-948)
3) ☐ Information Disclosure Statement(s) (PTO-1449 or PTO/SB/08) Paper No(s)/Mail Date _____.
4) ☐ Interview Summary (PTO-413) Paper No(s)/Mail Date. _____ .
5) ☐ Notice of Informal Patent Application (PTO-152)
6) ☐ Other: _____.

DETAILED ACTION

1. It would be of great assistance to the Office if all incoming papers pertaining to a filed application carried the following items:

1. Application number (checked for accuracy, including series code and serial no.).
2. Group art unit number (copied from most recent Office communication).
3. Filing date.
4. Name of the examiner who prepared the most recent Office action.
5. Title of invention.
6. Confirmation number (See MPEP § 503).

The amendment to fig.13 on page 25, lines 30-34 is not supported by the drawing fig.13 as originally filed.

Fig. 13 shows a 32-**inverse** Fourier transform 1315 as originally filed.(not 32- Fourier transform).

Claim Rejections - 35 USC § 112

2. The following is a quotation of the first paragraph of 35 U.S.C. 112:

> The specification shall contain a written description of the invention, and of the manner and process of making and using it, in such full, clear, concise, and exact terms as to enable any person skilled in the art to which it pertains, or with which it is most nearly connected, to make and use the same and shall set forth the best mode contemplated by the inventor of carrying out his invention.

3. Claims 24,26-28,32 are rejected under 35 U.S.C. 112, first paragraph, as failing to comply with the written description requirement. The claim(s) contains subject matter which was not described in the specification in such a way as to reasonably convey to one skilled in the relevant art that the inventor(s), at the time the application was filed, had possession of the claimed invention.

4. In claim 24 and 32, the particular limitation "a 32-inverse Fourier transformer that receives S and S* signal" is not supported by the specification as originally filed. The

specification page 17, lines 22-23, figs. 4 and 11 disclose –a 32-inverse **Fast** Fourier transformer block 407 and block 1117. (IFFT is not the same as IFT)(See attached definition of IFFT and FFT)

Allowable Subject Matter

5. Claims 25,29-31 and 33 are allowed.

Response to Arguments

6. Applicant's arguments filed 7/28/05 and 11/11/05 have been fully considered but they are not persuasive. First examiner will like to point out to the applicant's representative, that the examiner never said, "a fast Fourier transform is not a type of Fourier transform". Examiner tried to explain to the applicant representative that inverse **Fast** Fourier transform is not the same as inverse Fourier Transform. Examiner is sending with this office action a copy a definition of IFFT and IFT.

7. Any inquiry concerning this communication or earlier communications from the examiner should be directed to Temesghen Ghebretinsae whose telephone number is 571-272-3017. The examiner can normally be reached on Monday-Friday from 8 to 6. The examiner can also be reached on alternate .

If attempts to reach the examiner by telephone are unsuccessful, the examiner's supervisor, Jay Patel, can be reached on 571-272-2988. The fax phone number for the organization where this application or proceeding is assigned is 571-273-8300.

Information regarding the status of an application may be obtained from the Patent Application Information Retrieval (PAIR) system. Status information for

published applications may be obtained from either Private PAIR or Public PAIR. Status information for unpublished applications is available through Private PAIR only. For more information about the PAIR system, see http://pair-direct.uspto.gov. Should you have questions on access to the Private PAIR system, contact the Electronic Business Center (EBC) at 866-217-9197 (toll-free).

Temesghen Ghebretinsae
Primary Examiner
Art Unit 2637

TEMESGHEN GHEBRETINSAE
PRIMARY EXAMINER

T.G

12/13/05.

Fourier Transforms, DFTs, and FFTs

Latest revision, 01 October 2003, 4:20 p.m.

Fourier Transform (FT)

- The ***Fourier transform*** (FT) is a generalization of the Fourier series.
- Instead of sines and cosines, as in a Fourier series, the Fourier transform uses exponentials and complex numbers.
- For a signal or function f(t), the ***Fourier transform*** is defined as

FT
$$\boxed{F(\omega) \equiv \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt}$$

and the ***inverse Fourier transform*** is defined as

$$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega}$$

where

  - i is the imaginary unity number, defined as the square root of -1, i.e. [x]. Note that in the text, j is used instead of i.
  - ω is the range of angular frequencies associated with the signal, i.e. the frequency content of the signal.
- When working with time and the signal f(t), one is said to be working in the ***time domain***, and the variables are ***real***.
- When working with angular frequency and the Fourier transform $F(\omega)$, one is said to be working in the ***frequency domain***, and $F(\omega)$ is ***complex***.
- The FT can be thought of as an ***analog*** tool, since it is used for analyzing the frequency content of *continuous* signals.

Discrete Fourier Transform (DFT)

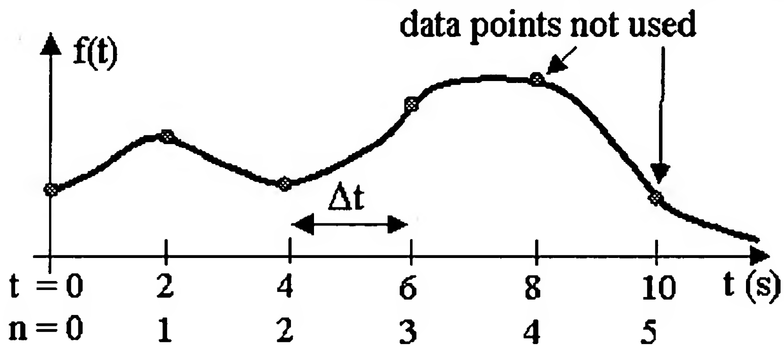
- One can define a Fourier transform for a *discrete* (digital) signal as well, called the ***discrete Fourier transform*** (DFT).
- The DFT can be thought of as a ***digital*** tool, since it is used for analyzing the frequency content of *discrete* signals.
- The discrete Fourier transform is defined as

FFT
$$\boxed{F(k\Delta f) \equiv \sum_{n=0}^{N-1} f(n\Delta t)e^{-i(2\pi k\Delta f)(n\Delta t)}} \text{ for } k = 0, 1, 2, ..., N-1.$$

Note that summation has been used in place of the integral since here *discrete* rather than *continuous* data are being examined.
- In the *time domain*, the relevant variables are defined below:
  - N = total number of discrete samples taken.
  - T = total sampling time.
  - Δt = time increment between samples. $\boxed{\Delta t = \frac{T}{N}}$.
  - f_s = the sampling frequency. $\boxed{f_s = \frac{1}{\Delta t} = \frac{N}{T}}$.
- Note that integers n and k in the above definition have values from 0 to N-1, not from 1 to N.
- As an example, suppose data are sampled discretely at a sampling frequency of 0.5 Hz. Starting at time t = 0, N = 4 samples are taken in T = 8 seconds. Here,

, and $f_s = \frac{1}{\Delta t} = 0.5$ Hz.

The discrete data used to calculate the DFT are those at t = 0, 2, 4, and 6 seconds. The data point at n = N, i.e. at t = T = 8 seconds, is not used, as the following plot indicates



- In general, unlike with Fourier series analysis, T in the discrete Fourier transform has nothing to do with the fundamental period of the original signal. In fact, the sampling may stop at a different *phase* in the signal than where sampling started, as in the time trace shown below.

  This leads to some inaccuracies when using discrete Fourier transforms, as discussed later.
- In most actual experimental measurements, the fundamental period of the signal is not even *known*; thus ***T is totally unrelated to anything inherent in the signal itself***. In some experimental measurements, the signal to be examined by a Fourier transform is not even purely periodic!
- In the *frequency domain*, the relevant variables are defined below:
  - Δf = the ***frequency increment*** for the output.

    .

    The frequency increment Δf can also be called the ***frequency resolution***.
  - $F(k\Delta f)$ = the discrete Fourier transform output, one complex value for each discrete frequency, which provides information about the relative contribution to the signal by each discrete frequency.
- The frequency increment, Δf, of a DFT is analogous to the ***fundamental frequency*** of a Fourier series in that the DFT provides information about the relative contribution of the ***harmonics*** of Δf, just as the Fourier series coefficients provide information about the relative contribution of the harmonics of the fundamental frequency.
  - For k = 1, $F(1\Delta f)$ is the DFT at the first harmonic frequency, Δf.
  - For k = 2, $F(2\Delta f)$ is the DFT at the second harmonic frequency, $2\Delta f$.
  - For k = 3, $F(3\Delta f)$ is the DFT at the third harmonic frequency, $3\Delta f$, and so on.
- Note, however, that unlike a Fourier series, ***this frequency increment, Δf, in general, has nothing to do with the frequency content or the fundamental frequency of the original signal***! Why? Because, Δf is simply 1/T, where T is an arbitrary time period (the total sampling time), as discussed above. This leads to some inherent errors associated with discrete Fourier transforms that will become clearer when examples are shown later.
- The ***Nyquist criterion*** comes into play here also. Namely, when sampling at some sampling frequency, f_s, ***one can obtain reliable frequency information only for frequencies less than $f_s/2$***. (Here, *reliable* means without *aliasing* problems.)
- Consider the DFT output. There are N output values, at discrete frequencies, i.e. $F(k\Delta f)$ for k = 0, 1, 2, ..., N-1. Since and f_s = N/T, one can easily calculate at what value of k the frequency $k\Delta f$ equals $f_s/2$. I.e.,

$k\Delta f = \frac{f_s}{2}$ when $k = \frac{f_s}{2\Delta f} = \frac{N/T}{2(1/T)} = \frac{N}{2}$.

- The result of this is that ***only half of the N DFT output values are useful***, i.e. only up to k = N/2, or frequency up to $f_s/2$. This is a direct result of the Nyquist criterion.
- For example, suppose one samples a signal for T = 4 seconds, at a sampling rate of f_s = 100 Hz.
  - How many samples are taken? Answer: $N = Tf_s$ = (4 s)(100 Hz) = 400 samples.
  - How many *useful* DFT output values are obtained? Answer: N/2 = 200.
  - What is the frequency resolution, Δf? Answer: $\Delta f = \frac{1}{T} = \frac{1}{4\text{ s}} = 0.25\text{ Hz}$.
  - What is the maximum frequency for which the DFT output is reliable? Answer: $f_{max} = f_s/2$ = (100 Hz)/2 = 50 Hz. Or, $f_{max} = \frac{N}{2}\Delta f = \frac{400}{2}(0.25\text{ Hz}) = 50\text{ Hz}$.
  - This maximum frequency is also called the ***folding frequency***, which shall be designated as $f_{folding}$ or f_N (the textbook uses the notation f_N for the folding frequency.)

Fast Fourier Transform (FFT)

- The ***fast Fourier transform*** is simply a DFT that is faster to calculate on a computer. see DFT
- All the rules and details about DFTs described above apply to FFTs as well.
- For most FFTs, the computer algorithm restricts N to a power of 2, such as 64, 128, 256, and so on. However, some of the newer FFT algorithms do not have such a restriction. (For example, the FFT used in LabVIEW does not have this power of 2 restriction.)
- It is not the goal of this learning module to teach the details of how to efficiently calculate an FFT. Instead, the goal here is to teach the student how to use and interpret FFTs, and how to set up the parameters so as to achieve adequate frequency resolution while minimizing problems such as leakage, which will be discussed below.
- The output, $F(k\Delta f)$, of an FFT subroutine is a series of complex numbers, one for each discretely sampled data point, representing each discrete frequency, only half of which are useful because of the Nyquist criterion.
- It is the ***magnitude*** or ***amplitude*** of the complex number generated by the FFT that is actually used to compare the relative importance of the various frequencies. Recall that for some complex number z = x + iy, where x is the real part and y is the imaginary part, the magnitude of z is given by $|z| = \sqrt{x^2 + y^2}$.
- Note that the magnitude of a complex number is called the ***modulus***; in Microsoft Excel, the function to calculate modulus is ***IMABS***.

Frequency Spectrum

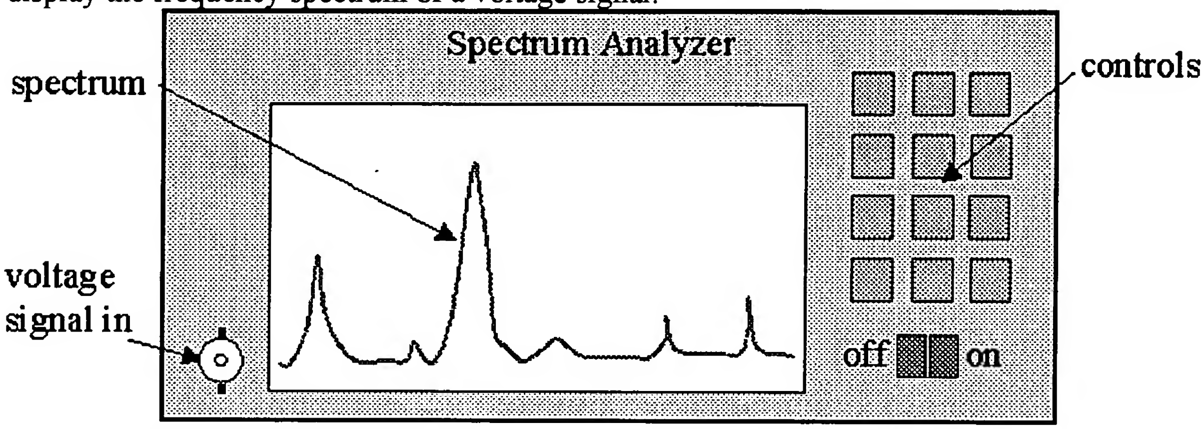
- A plot of the magnitude of the FFT output, $|F|$, as a function of frequency, f, is called the ***frequency spectrum***. Some authors call this the ***amplitude spectrum*** or the ***energy spectrum***, but the dimensions of the vertical axis may not always be consistent with that implied by these names, so the author prefers to call the plot simply a frequency spectrum.
- Over a given frequency range, the amplitude defined in this way indicates the ***relative importance*** of that frequency range to the signal. A frequency spectrum plot formed from an FFT is analogous to the harmonic amplitude plot formed from a Fourier series.

Usefulness of the FFT

- The fast Fourier transform (FFT) is extremely useful in analyzing ***unsteady measurements***, because ***the frequency spectrum from an FFT provides information about the frequency content of the signal***.
- Examples come from nearly every area of engineering, such as vibrations, where one needs to know the frequency content of the vibration; fluid flow, where one needs to know the frequency content of the turbulent fluctuations; and acoustics, where one needs to know the frequency content of a sound signal, to mention just a few.

Spectrum Analyzers

- There are laboratory instruments called ***spectrum analyzers*** which are designed to calculate and display the frequency spectrum of a voltage signal.



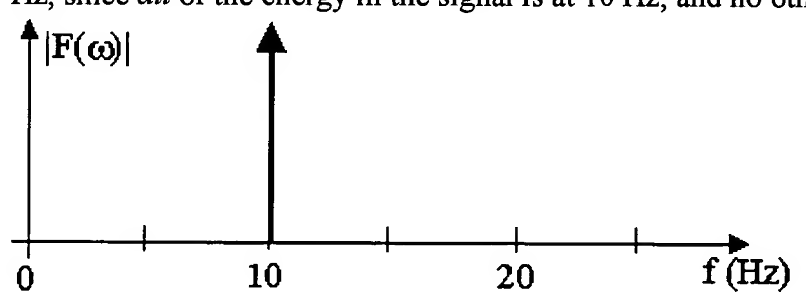
- Internally, an FFT is calculated.
- In the lab, students will generate a ***virtual spectrum analyzer*** using LabVIEW.

FFT Examples

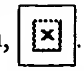
Examples are shown to illustrate some of the results and problems encountered when using FFTs.

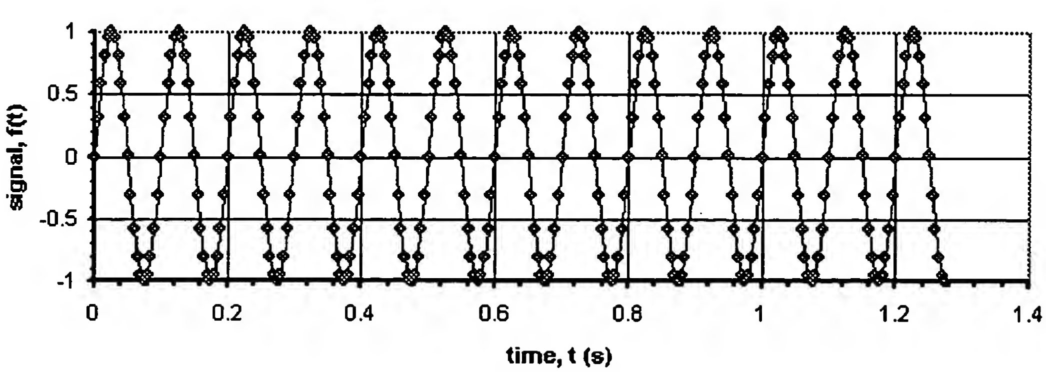
A pure sine wave

- Consider first the FFT of a ***pure sine wave***. Suppose the signal is a 10 Hz sine wave with peak-to-peak amplitude -1 to 1 volt, i.e. $f(t) = \sin(2\pi \cdot 10t)$.
- The ***ideal Fourier transform*** would have a spike of magnitude 1 Volt at a frequency of exactly 10 Hz, since *all* of the energy in the signal is at 10 Hz, and no other frequency contains any energy.

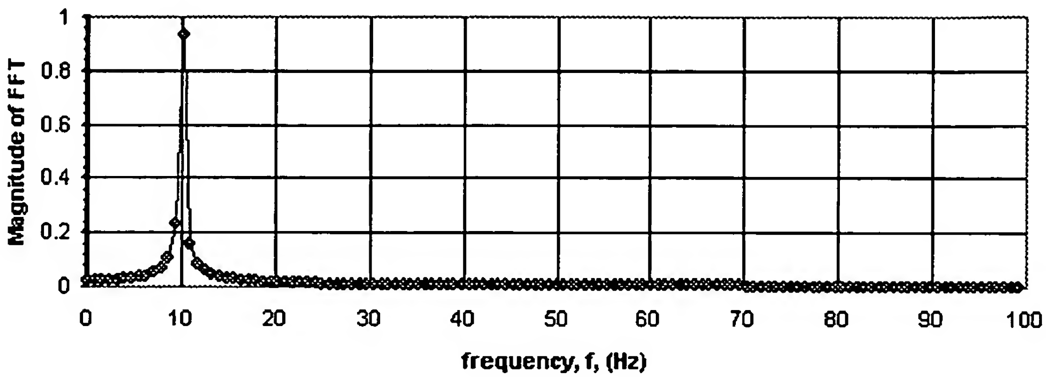


- Unfortunately, the actual frequency spectrum generated by an FFT of this signal will not be that clean, however, but depends on a number of factors, such as
  - sampling rate, f_s
  - number of samples, N
  - total time of data collection, T, where $T = N/f_s$

  - o the windowing function used (to be explained later)
- Note: The frequency axis of the FFT goes from 0 to $f_s/2$, and the output is at discrete frequencies with an interval determined by the frequency resolution, .
- Most spreadsheets have a built in FFT function. In fact, the frequency spectra shown in this learning module were generated and plotted using ***Microsoft Excel***. In Excel, the FFT can be calculated and plotted as follows:
  - o A column of discrete sample times is created.
  - o Beside that column, a column representing the discrete signal is created.
  - o The mouse is clicked on Tools-Data Analysis-Fourier Analysis, and OK. (Note: If the data analysis tool is not available in Excel, it must be added in by Tools-Add-Ins- Analysis ToolPak, and OK. After this, Data Analysis should appear as an option in Tools.)
  - o The input range of the signal to be analyzed is entered, and a column is chosen for the output by defining the top cell of the column for the top of the output. OK. A column of complex numbers will be written to the spreadsheet, beginning with the cell selected for writing the output.
  - o Since the output of the FFT is a column of complex numbers, each of these must be converted to an amplitude by using the IMABS function. The amplitudes are calculated in a separate column.
  - o Each of these amplitudes must be divided by (N/2) so that the corrected amplitude corresponds directly to the amplitude of the signal, as will be shown later.
  - o In addition, the *first* amplitude (at f = 0 Hz) must also be divided by 2 so that the amplitude corresponds properly to the DC offset of the signal.
  - o Note that only *half* of these amplitudes are useful for the frequency spectrum, because of the Nyquist criterion.
  - o Another column is created for the discrete frequencies, spaced by frequency interval Δf.
  - o The values in the amplitude column are then plotted as a function of frequency in the usual manner. The result is the frequency spectrum.
- An example is shown in Appendix A.2 of the text. One caution when using Excel - when a parameter is changed so that the input signal changes, ***the FFT output in Excel is not automatically updated when the time trace is updated. The user must re-run the FFT macro every time the time trace changes.***
- To illustrate FFTs, consider a simple 10 Hz sine wave with amplitude = 1.
- As a starting point, f_s was chosen to be 200 samples per second, and N was chosen to be 256. In other words, the 10 Hz sine wave was sampled at 200 Hz for 1.28 seconds ($T = N/f_s = 256/200 = 1.28$). Both the time trace (the signal in the time domain) and the frequency spectrum (the magnitude of the FFT output in the frequency domain) are shown below.
- First the time trace, showing the discretely sampled data:

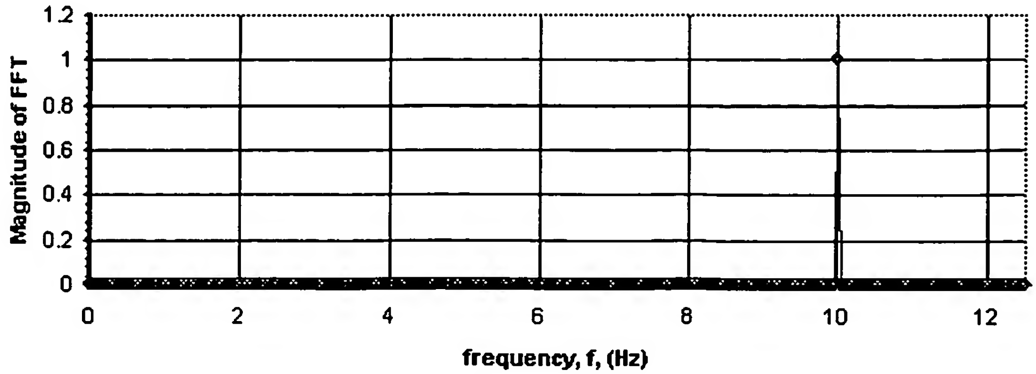


Second, the frequency spectrum for N = 256 and $f_s = 200$. The frequency resolution for the spectrum is = 1/1.28 = 0.781 (to three significant digits):



- The following points about the above plot are to be noted:
  - Although the frequency spectrum correctly shows a spike at 10 Hz, the spike is not infinitesimally narrow. In fact, it appears from the frequency spectrum that there is a significant component of the signal at frequencies *near* 10 Hz (specifically within about 5 Hz to either side of 10 Hz). This unphysical error in the FFT is called ***leakage***, which appears when the discrete data acquisition does not stop at exactly the same phase in the sine wave as it started. In principle, if an infinite number of discrete samples are taken, leakage would not be a problem. However, any real data acquisition system performing FFTs uses a finite (rather than infinite) number of discrete samples, and there will always be some leakage.
  - The maximum amplitude of the FFT is not exactly 1 (in fact it is less than 1), even though the amplitude of the original signal was exactly 1. This is another consequence of leakage - namely some of the energy at 10 Hz is erroneously distributed among frequencies near 10 Hz, thereby reducing the calculated amplitude at 10 Hz.
  - The frequency range is from 0 to 100 Hz, half of the sampling frequency, as discussed above. This is a consequence of the Nyquist criterion.
- How can leakage be reduced? Will a higher sampling rate help? Let us repeat this example, but with a higher sampling rate.
- The same number of samples (256) of this sine wave are now taken at 1000 Hz instead of 200 Hz. All other parameters remain the same. The time trace is shown below:
- The frequency spectrum corresponding to this time trace is shown below:
- Some observations about these results:
  - Higher sampling frequency has vastly improved resolution of the sine wave; there are now 100 data points per wavelength.
  - However, increasing the sampling frequency did *not* improve the frequency spectrum at all!
  - In fact, the output frequency spectrum has even *poorer* frequency resolution. This can be clearly predicted by the above equations. Namely, the frequency resolution for this case is = approximately 3.91 Hz, whereas that of the previous case (f_s = 200 Hz) was approximately 0.781 Hz.
  - Furthermore, the peak amplitude of the frequency spectrum is now about 0.66, significantly smaller than the known signal amplitude of 1. This is due to leakage, which is even worse in

  this case because of the poorer frequency resolution.
  - Even worse, the peak amplitude of the frequency spectrum occurs at a frequency of 11.72 Hz instead of the known 10 Hz. This is also a result of the poor frequency resolution.
  - One can conclude: ***The only thing gained by increasing the sampling frequency is the corresponding increase in the maximum frequency or folding frequency of the frequency spectrum.***
- For this example, a 10 Hz sine wave, a 1000 Hz sampling rate is *overkill*. In fact, the original 200 Hz sampling rate is also higher than necessary. By the Nyquist criterion, since the maximum frequency in the signal is 10 Hz, the minimum required sampling rate to avoid aliasing is 20 Hz.
- The signal was again sampled (256 samples as before), but this time at a sampling frequency of only 25 Hz. The frequency spectrum is shown below:
- Some comments about this frequency spectrum:
  - This time, the frequency resolution is much better (0.0977 Hz).
  - While there is still some leakage, it is only significant within about 1 Hz to the right or left of the peak at 10 Hz.
  - The maximum amplitude is about 0.75, and occurs at 9.96 Hz - much closer to the true frequency of 10 Hz, due to the greater resolution.
  - This illustrates a very significant, yet often unappreciated fact: ***Higher sampling rate is not always better***!
- Can one improve the FFT results any further? The answer is yes, but at the cost of more computing time. The previous case was repeated, except the number of samples was doubled to 512. Everything else remained the same, i.e. the sampling frequency was 25 Hz, and the same sine wave was sampled. The results are shown below:
- Some comments:
  - Compared to the previous case, the frequency resolution has improved by a factor of two because the number of discrete data points has increased by a factor of two (for the same sampling frequency).
  - The peak in the spectrum at 10 Hz is much narrower, with much reduced leakage.
  - The peak amplitude is now 0.934 at a frequency of 10.01 Hz. This is *much* closer to the known exact amplitude of 1 at 10 Hz. The improvement is due to greater frequency resolution and reduced leakage.
- A final comment needs to be made about FFTs and frequency spectra. Namely, it is theoretically possible to obtain a *perfect* FFT (i.e. one without any leakage) from sampled data.
- However, this is only possible if the sampled data begin and end at the *same phase* of the signal.
- To illustrate, the above example was repeated one more time, but this time with a sampling frequency of exactly 25.6 Hz.
- This sampling frequency was chosen such that there is an integral number of wavelengths in the sampled data set, so that the sampled signal starts and ends at exactly the same phase. In this particular case, $f_s = 25.6$, $N = 512$, $T = N/f_s = 20$ s exactly. This T corresponds to exactly 200 complete cycles for our 10 Hz sine wave. The sampled signal starts and ends at zero phase.
- The resulting frequency spectrum is shown below:



- Some comments:
  - The peak amplitude is exactly 1 V, and occurs at exactly 10 Hz.
  - The FFT is perfectly correct because there is *no leakage* in this case.
  - The reader must keep in mind that this "perfect FFT" was possible only because we knew the frequency of the original signal, and selected the sampling frequency appropriately. In real-life laboratory situations, the frequency is not known in advance (otherwise, an FFT would not even be necessary!).
  - In conlusion, although a perfect FFT is possible, it will never be encountered in actual laboratory situations.
- This leads us into the next topic, i.e. windowing.

Windowing

- Can leakage be reduced even further without increasing the number of samples? The answer is yes, through a process called ***windowing***.
- As mentioned above, the cause of leakage is that the signal starts and ends at different phases in the cycle. In most experiments, the sampling rate and number of samples will be such that this cannot be avoided. However, if the amplitude of the signal at both ends (near t = 0 and near t = T) is reduced to zero in a smooth fashion, the end effects will not be as serious. This is the fundamental concept of windowing.
- In practice, the signal is multiplied by a ***windowing function***, which starts at zero at t = 0, increases smoothly to 1 at t = T/2, and then decreases smoothly again to zero at t = T.
- Many such windowing functions have been invented, but the most common one is the ***Hanning window***, created by a simple cosine wave. The ***Hanning window function***, u(t) is defined as
  $$\boxed{u(t) \equiv \frac{1}{2}\left[1 - \cos\left(\frac{2\pi t}{T}\right)\right] \quad \text{for } 0 \le t \le T}$$
- As an example, return to the original 10 Hz sine wave funtion, f(t). For this example, the sampling frequency is 90 Hz, and N = 256. The total sampling time is thus $T = N/f_s = 2.8444...$ s. Plotted below are f(t), u(t), and their product f(t)*u(t):

  Notice that the modified time trace, f(t)*u(t), trails off to zero at both ends (t = 0 and t = T). The middle region of the time trace is relatively unaffected by the windowing function. In essence, ***the windowing operation emphasizes the middle portion of the time trace, and de-emphasizes the ends***.
- Does this reduce leakage? Yes, as can be seen in the frequency spectra below. Note that since the sampling frequency is 90 Hz in both cases, the maximum frequency or folding frequency is half of this, i.e. 45 Hz.

Shown above is the frequency spectrum of the original signal, f(t), and on the same plot the frequency spectrum of the modified (windowed) signal, f(t)*u(t). The modified FFT clearly has less leakage, although leakage still exists to some degree.

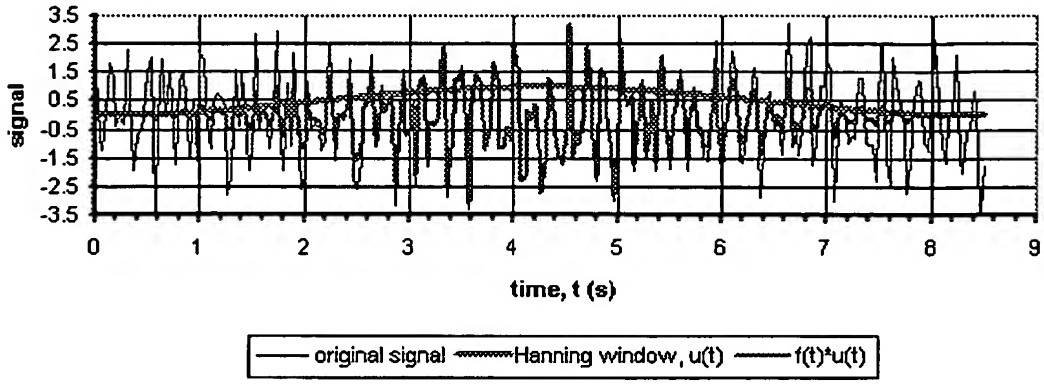
- The Hanning windowing operation causes a loss of information near the end points of the signal. This leads to an overall reduction in the amplitude of the frequency spectrum. To compensate, it turns out that the amplitude of the FFT must be multiplied by a factor of , which brings the amplitude back up to the proper level.
- ***The factor of has been included in the above plot.*** As can be seen on the plot, the peak amplitude of the modified spectrum is approximately the same as that of the original spectrum, but leakage has indeed been reduced.
- As explained above, this is *still* lower than the known input amplitude of 1.0 volts because of leakage.
- The bottom line: ***Windowing reduces but does not totally eliminate leakage.***

Frequency spectrum of an unknown signal

- When the frequency content of the signal is already known, as in the above sine wave example, it is easy to choose the sampling frequency as some frequency larger than twice the maximum frequency of the signal (using the Nyquist criterion).
- However, what should be done when trying to determine the frequency content of an *unknown* signal (the more typical laboratory situation), when the frequency content is not known? There are a few things one can do:
  - One can sample at several different sampling frequencies and compare the resulting frequency spectra. If amplitude peaks appear at *different* frequencies, one can be sure that aliasing errors are occurring. In this case, the sampling frequency must be increased until the peaks in the spectra do not change, and the aliased peaks can be consistently explained. An example is provided below.
  - To be sure that the sampling frequency is high enough, the frequency spectrum should also drop off toward zero amplitude near the maximum frequency or folding frequency.
  - To avoid aliasing errors entirely, one can ***filter*** the signal before calculating the FFT.
    - A ***low-pass filter*** is used to eliminate aliasing, and is often called an ***anti-aliasing filter***.
    - A low-pass filter lets low frequency components of the signal pass through, but cuts off frequency components above some cut-off frequency.
    - Ideally, the cut-off frequency should be one-half of the sampling frequency to avoid aliasing (Nyquist criterion). Unfortunately, real-life low-pass filters do not cut off high frequencies abruptly; instead, the attenuation of higher frequencies falls off rather slowly with frequency. Because of this, most real-life data acquisition systems will employ a cut-off frequency several times smaller than the sampling frequency for proper anti-aliasing.
    - Low-pass filters will be discussed in detail later in the semester.

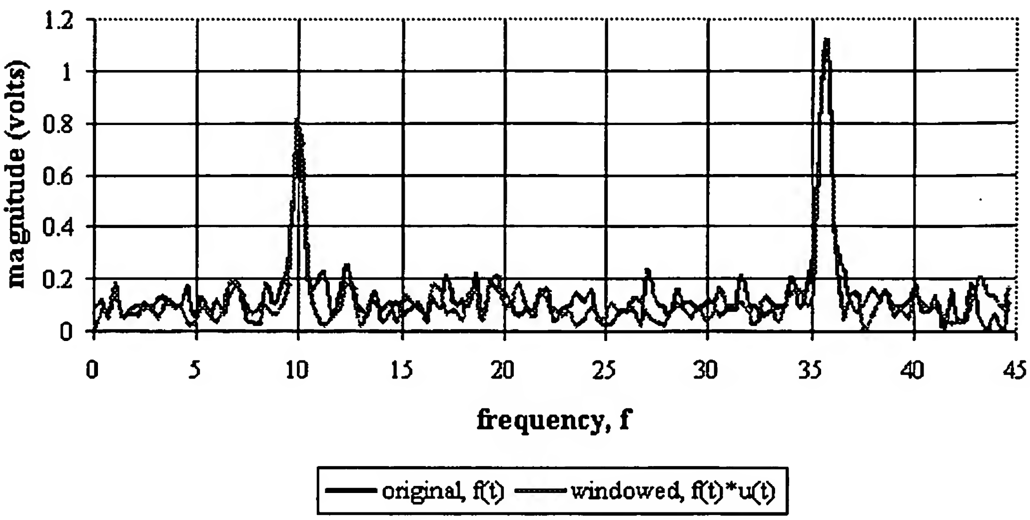
Example

- To illustrate how to find the frequency content of an unknown signal, a signal with more than one frequency component was generated, along with some random noise.
- The first case is sampled at $f_s = 30$ Hz, with $N = 256$. The time trace is shown below:

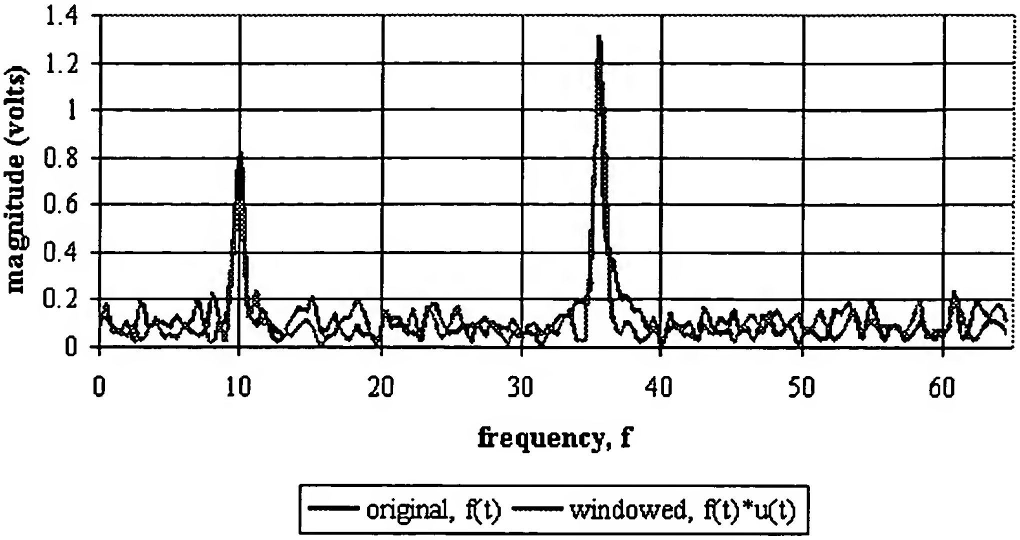


Note that the function has been multiplied by the Hanning window function, as previously.

- The corresponding frequency spectra for the original signal and for the windowed signal are shown below:
- Note that the factor of has been included in the above plot, as in the previous example.
- Clearly, there is a strong 10 Hz component of this signal, along with an even stronger component at around 5.7 Hz. The rest of the frequency spectrum shows random noise.
- Windowing has reduced some of the leakage near the peaks at 10 Hz and 5.7 Hz, but has also changed the rest of the spectrum somewhat. Why? Because portions of the signal near t = 0 and near t = T have been effectively *removed* by windowing. In other words, ***because of windowing, some of the information in the original signal has been lost***. Nevertheless, windowing is nearly always advisable.
- Can this spectrum be trusted? One quick way to tell is to sample at a higher frequency. The next case is for f_s = 66 Hz. (N is kept at 256 for all of these examples.) The frequency spectrum is shown below:
- The peak at 10 Hz is still present, but the one at 5.7 Hz is gone. Instead is a peak at 30.3 Hz! What has happened? Clearly, ***aliasing errors occurred in the first case***, and the sampling rate may *still* not be high enough.
- One can examine the first two cases to see if they are consitent:
  - Sampling at 66 Hz yields a frequency component at 30.3 Hz.
  - Since 66 is greater than 2(30.3), it is possible that this represents a legitimate frequency component of the signal.
  - If this *is* legitimate, we should be able to predict the aliased frequency from the first case. Let's check:
    - Sampling at 30 Hz yielded a frequency component at 5.7 Hz.
    - If the actual frequency component of the signal was 30.3 Hz, one would predict an aliasing frequency of 30.3-30.0 = 0.3 Hz when sampling at 30 Hz.
    - However, this is *not* consistent with the observed frequency of 5.7 Hz when sampling at 30 Hz.
  - Thus, because of the inconsistencies, one can conclude that the sampling rate of 66 Hz is *still* not high enough.
  - Sampling must be increased to an even higher frequency to resolve the problem.
- The next case is for an even higher sampling frequency of 90 Hz. The spectrum is shown below:



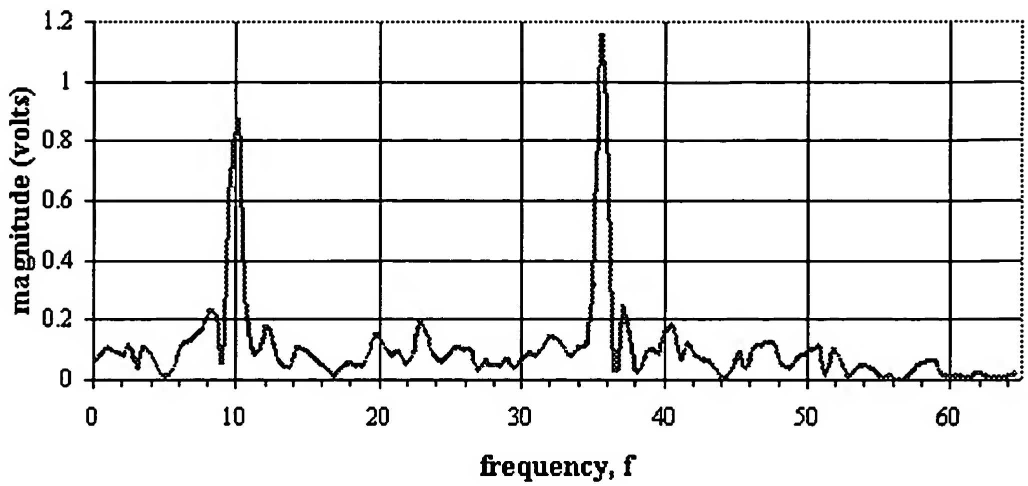
- This time, the peak at 10 Hz remains, but the other peak is at about 35.7 Hz. Apparently, the previous peaks at 5.7 Hz and 30.3 Hz were not real, but were falsely generated due to aliasing. The peak at 10 Hz is real.
- This indicates that f_s = 66 Hz was not a high enough sampling frequency.
- Is 90 Hz a sufficient sampling rate for this signal? Is aliasing still a problem? To check, one can examine the first three cases to see if they are consitent:
  - Sampling at 90 Hz yields a frequency component at 35.7 Hz.
  - Since 66 is greater than (2/3)(35.7) but less than 2(35.7), a legitimate frequency component at 35.7 Hz would be aliased to 66.0-35.7 = 30.3 Hz when sampling at 66 Hz.
  - This is consistent with the frequency observed when sampling at 66 Hz.
  - Since 30 is greater than (2/3)(35.7), but less than 2(35.7), a legitimate frequency component at 35.7 Hz would be aliased to 35.7-30.0 = 5.7 Hz when sampling at 30 Hz.
  - This is consistent with the frequency observed when sampling at 30 Hz.
  - Thus, since everything is now consistent, one can conclude that the sampling rate of 90 Hz *is* high enough.
  - The bottom line: The signal contains:
    - a component at 10 Hz, with an amplitude greater than 0.8 volts (the exact amplitude cannot be known because leakage has reduced the height of the peak at 10 Hz).
    - a component at 35.7 Hz, with an amplitude greater than 1.3 volts (the exact amplitude cannot be known because leakage has reduced the height of the peak at 35.7 Hz).
    - some noise at all frequencies (random noise).
- As a final check, another case was run at a sampling frequency of 130 Hz, the spectrum of which is shown below:



- Now, finally, the peaks have remained at the same frequencies (10 Hz and 35.7 Hz). One can be fairly confident now that the signal is being sampled at high enough frequency to avoid aliasing. The two major peaks in the frequency spectrum have been correctly captured.
- One problem still remains - there is still some noise, even at frequencies higher than 60 Hz. This problem can be overcome with filtering.

Filtering

- So far, this signal was not filtered in any way. Since there is still non-zero amplitude around the folding frequency of 65 Hz, the sampling frequency of 130 Hz may still not have been high enough to capture all frequencies in the signal.
- However, the high frequency energy is merely noise, and is really of no interest in this particular problem.
- One way to remove the high frequency noise in this example is to low-pass filter the signal with a cut-off frequency somewhat above the maximum frequency of interest in the problem, i.e. 35.7 Hz, and then sample at a frequency a few times higher than this cut-off frequency, depending on the type of low-pass filter used.
- To illustrate, the signal was filtered with a eighth-order low-pass Butterworth filter, with a cut-off frequency of 50 Hz. Data were sampled at 130 Hz and N = 256 as previously. The frequency spectrum (after Hanning windowing) is shown below:



Note that the FFT of the original signal is not shown on this plot so that the effects of the filter can be clearly seen.

- Notice how the magnitude of the frequency spectrum drops off beyond the cut-off frequency, as desired.
- Notice also that the amplitude of the peak at 35.7 Hz has also been reduced somewhat. This is because the Butterworth filter actually starts attenuating frequencies even smaller than the cut-off frequency. (In addition, due to the random nature of the signal, the signal sampled for this FFT was not the identical signal that had been sampled for the previous FFTs.)
- Only now can one completely trust the results of the frequency spectrum. In this example, one can state confidently the following:
    - There is a frequency component of the signal at 10 Hz, with an amplitude greater than 0.8 V.
    - There is an even stronger frequency component at 35.7 Hz, with an amplitude greater than 1.3 V.
    - There is random noise over the rest of the frequency range.
    - There *may* be some other minor peaks at frequencies above 65 Hz, but these cannot be identified, and are not of interest; they are buried in the background noise.
- This example illustrates some of the frustrations and problems associated with FFT spectral analysis. ***Users must be very careful, or the frequency spectrum may not yield the correct frequency content of the signal***!

Take a quiz on the material in this learning module.
